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THE PROBLEM OF FDM EXPLICIT SCHEME STABILITY

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The problem of numerical schemes stability is closely associated with a numerical error. The FDM scheme is stable when the errors made at one time step of the calculation do not cause the errors to increase as the computations are continued. If, on the contrary, the errors grow with time the numerical scheme is said to be unstable. The stability of numerical schemes can be investigated by performing von Neumann stability analysis. According to this theory, the approximation error carried by $\theta_{i,j}^f$ (2D problem) at every node of space (i, j) and time f is assumed to have a wave form with the wave numbers denoted by s_1, s_2 and the amplitude by δ:

$$\theta_{i,j}^f = \delta^f \exp\left[I\left(s_1 x_i + s_2 y_j\right)\right], \quad I = \sqrt{-1}$$

As time progresses, to assure convergence, the amplitude of approximation error must be less than unity, i.e. $\left|\theta_{i,j}^f\right| < 1$

Formulation of the problem

2D Fourier equation:

$$(x,y) \in \Omega: \quad \frac{\partial T(x,y,t)}{\partial t} = a\nabla^2 T(x,y,t)$$

The boundary conditions on the external surface of the system in a general form:

$$(x,y) \in \Gamma: \quad \Phi\left[T(x,y,t), \frac{\partial T(x,y,t)}{\partial n}\right] = 0$$

The initial condition:

$$t = 0: \quad T(x,0) = T_0$$

Cartesian co-ordinate system covered by the rectangular differential mesh with steps h and k. $P_{i,j}$ - the central point of the star is denoted.

The FDM equation for node $P_{i,j}$ (explicit scheme):

$$\frac{T_{i,j}^f - T_{i,j}^{f-1}}{\Delta t} = a\left(\frac{T_{i+1,j}^{f-1} - 2T_{i,j}^{f-1} + T_{i-1,j}^{f-1}}{h^2} + \frac{T_{i,j+1}^{f-1} - 2T_{i,j}^{f-1} + T_{i,j-1}^{f-1}}{k^2}\right)$$

or

$$T_i^f = \left(1 - \frac{2a\Delta t}{h^2} - \frac{2a\Delta t}{k^2}\right)T_i^{f-1} + \frac{a\Delta t}{h^2}\left(T_{i+1,j}^{f-1} + T_{i-1,j}^{f-1}\right) + \frac{a\Delta t}{k^2}\left(T_{i,j+1}^{f-1} + T_{i,j-1}^{f-1}\right)$$

Solution to the problem

$$\delta^f \exp\left[I\left(s_1 x_i + s_2 y_j\right)\right] = \left(1 - \frac{2a\Delta t}{h^2} - \frac{2a\Delta t}{k^2}\right)\delta^{f-1}\exp\left[I\left(s_1 x_i + s_2 y_j\right)\right] +$$
$$\frac{a\Delta t}{h^2}\delta^{f-1}\left\{\exp\left[I\left(s_1 x_{i+1} + s_2 y_j\right)\right] + \exp\left[I\left(s_1 x_{i-1} + s_2 y_j\right)\right]\right\} +$$
$$\frac{a\Delta t}{k^2}\delta^{f-1}\left\{\exp\left[I\left(s_1 x_i + s_2 y_{j+1}\right)\right] + \exp\left[I\left(s_1 x_i + s_2 y_{j-1}\right)\right]\right\}$$

or dividing by $\delta^{f-1}\exp\left[I\left(s_1 x_i + s_2 y_j\right)\right]$

$$\delta = \left(1 - \frac{2a\Delta t}{h^2} - \frac{2a\Delta t}{k^2}\right) + \frac{a\Delta t}{h^2}\left[\exp\left(Is_1 h\right) + \exp\left(-Is_1 h\right)\right] +$$
$$\frac{a\Delta t}{k^2}\left[\exp\left(Is_2 k\right) + \exp\left(-Is_2 k\right)\right]$$

$$\delta = \left(1 - \frac{2a\Delta t}{h^2} - \frac{2a\Delta t}{k^2}\right) + \frac{2a\Delta t}{h^2}\cos\left(s_1 h\right) + \frac{2a\Delta t}{k^2}\cos\left(s_2 k\right)$$

$$\delta = 1 - \frac{2a\Delta t}{h^2}\left[1 - \cos\left(s_1 h\right)\right] - \frac{2a\Delta t}{k^2}\left[1 - \cos\left(s_2 k\right)\right]$$

because $1 - \cos\alpha = 2\sin^2\left(\frac{\alpha}{2}\right)$, therefore

$$\delta = 1 - \frac{4a\Delta t}{h^2}\sin^2\frac{s_1 h}{2} - \frac{4a\Delta t}{k^2}\sin^2\frac{s_2 k}{2}$$

Conclusions

The condition $\left|\theta_{i,j}^f\right| < 1$ leads to the system of inequalities $\frac{4a\Delta t}{h^2}\sin^2\frac{s_1 h}{2} + \frac{4a\Delta t}{k^2}\sin^2\frac{s_2 k}{2} > 0$ and $1 - \frac{2a\Delta t}{h^2}\sin^2\frac{s_1 h}{2} - \frac{2a\Delta t}{k^2}\sin^2\frac{s_2 k}{2} > 0$

The first of them is the unconditional inequality, while the worst situation in the case of the second inequality takes place when

$\sin^2\frac{s_1 h}{2} = 1$, $\sin^2\frac{s_2 k}{2} = 1$ and then $1 - \frac{2a\Delta t}{h^2} - \frac{2a\Delta t}{k^2} > 0$

In this way the well-known stability condition for the linear parabolic equations is found.